CONTRACT LAW AND THE VALUE OF A GAME

BY

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ABSTRACT

For the special case of games with linearly transferable utility, a treatment preserving the main features of the controversial treatment in the author's doctoral dissertation is newly derived from a model for negotiation and play that is more elaborate than most such models. The crucial point of the derivation is that the author's special bargaining theory is not needed; the usual Zeuthen-Nash-Harsanyi bargaining theory gives the same result. The main novelty in the model that makes this possible is the replacement of customary informal uses of "enforceable agreements" by explicit contract law. The problems of contract law for cooperative games seem to be very complex, and the present work makes only a bare beginning on them. A characteristic function and value are derived.

Introduction. This paper presents a value for certain n-person cooperative games (viz. those with linearly transferable utility) which, though technically new, could have been derived by trivial steps from two different places in the literature. There are two main points to the paper. Background: my previous work in cooperative games [4, 5] begins by changing utility theory and goes forward on radically different lines from anyone else's work; and, sound or not, it has not been followed. Main point 1: here I concede almost everything to the opposition, using their utility theory, their (Zeuthen-Nash-Harsanyi [1]) bargaining theory, and (though this is agreement, not concession) Shapley's formula, which can be based on a model due to Harsanyi [2], for deriving a value from a characteristic function. Nevertheless the value obtained is consistent with my previous scattered remarks [4, 5] on what a value should do, and inconsistent with the evaluations most recently proposed by Harsanyi [3], Selten [9], and Shapley [11]. The crucial turn of course lies under the word "almost". I depart from the custom of treating the coalition as a sort of religious order acting like a single person. I introduce a simple explicit contract law. It is actually too simple; for playing a game cooperatively, one would want a more sophisticated law, and its development appears to involve knotty problems. But the simple contracts used here seem to suffice for evaluation, in much the same way that ordinary mixed strategies suffice for a matrix game, because the means proposed secure certain results regardless of what the other players do.

Main point 2: for the games considered, one can speak of "the" other value, for Harsanyi [3], Selten [9], and Shapley [11] assign the same values to these games.

Received December 20, 1965, in revised form June 22, 1966.

The point is that the other value, to speak picturesquely, swindles certain players. Obviously there is room for considerable controversy about such a point. At any rate, the middle section of this paper concentrates on two three-person games which we evaluate differently. Selten's derivation of a value and other considerations lead us to examine a couple of related games.

I have profited from a confrontation with Selten on this question, and shall try to indicate some of his points about the examples. But the Selten value seems to require support by a theory at least as detailed as the simple theory given in section 1 below taking account of contract law. Toward that, Selten has made one point quite clear: he follows Schelling [8] in stressing unilateral commitments and feels they should have a legal standing if my contracts do. Lacking a detailed development, one can still note that if commitments are to have the force suggested by Schelling, there must be possible legal proceedings compelling "specific performance", the actual carrying out of promised actions. In contrast ,I propose to exclude specific performance, securing contracts only by penalty clauses for pecuniary (utility) indemnities.

Section 3 of the paper has the only non-trivial theorems, but is much the least interesting; it gives some results on "games with infinitely many players".

The ideas of this paper were worked out at the 1965 Jerusalem Game Theory Workshop, sponsored by The Hebrew University and the Israel Academy of Sciences and Humanities. I am much indebted to J. C. Harsanyi, L. S. Shapley, Martin Shubik and R. M. Thrall for constructive criticism there. The writing was supported by the National Science Foundation.

1. **Value.** For the simplest definition of the proposed value, we consider games in normal form. Though for most game theory, and for everything in this paper, the normal form suffices, we shall usually find it convenient to speak of the extensive form. If Γ is an n-person finite game in normal form with payoff function h, define $h^{-}(x_1, \ldots, x_n)$ for each (pure) strategy n-tuple $\xi = (x_1, \ldots, x_n)$ by $h_i^{-}(\xi) = h_i(\xi) + 1/n(m - \Sigma h_j(\xi))$, where m is the joint maximum $\max \Sigma h_i(\xi)$. Let Γ^{-} be the game derived from Γ by replacing the payoff h with h^{-}. We may follow an unpublished manuscript of Harsanyi and call Γ^{-} the *upraised game.* It is evidently constant-sum. We define the *upraised characteristic function* $v^{-} = v_{\Gamma}^{-}$ of Γ as the Neumann-Morgenstern characteristic function of Γ^{-} and the *contract value* ϕ^{-} of Γ as the Shapley value of Γ^{-}.

What does ϕ^{-} evaluate? To begin with, there is substantial agreement on the Shapley value for constant-sum transferable-utility games. As far as I know, none of the numerous values for more or less general games that have been proposed since Shapley's original paper [10] has differed on these games. A number of arguments leading to it are known. Note Shapley's axiomatic argument, adapted to the constant-sum setting in [4], and note Harsanyi's bargaining model [2].

Accordingly, we shall here take the step from v^- to ϕ^- as established, and examine the passage from Γ to v^-.

A specification of the assumptions involved that would be adequate for, say, the designer of a game-theoretic experiment, would be extremely long. Roughly, we need the usual assumptions after von-Neumann-Morgenstern [7], amplified by explicit legal assumptions (see below), and supplemented by a symmetry or democracy assumption to the effect that the players recognize each other as peers. That last assumption enters in two ways: in the (standard) bargaining theory to be applied to the model, and also in the structure of the model, where we seem to need a special meeting of all players. Of course no positive or negative concern of the players with each other's welfare is implied. In Zeuthen's and in Harsanyi's presentation of the bargaining theory [1], a player's resistance to a concession is supposed to depend monotonically on the ratio of the cost (to him) of the concession to the cost (to him) of a breakdown of negotiations, and by the same rule for all players. As for the special meeting of all players, it serves to conclude indefinitely long negotiations which *may* have already determined the outcome; but players considering whether to delay or not in the previous negotiations have the definite prospect of a last grand meeting, into which they will go with the support derived from previous agreements or the independence secured by previous disagreements, whichever they prefer.

Concerning the legal apparatus of the model, some preliminary remarks. Indisputably one wants contracts to be sufficiently definite so that a trial court can in a finite time determine whether the contract has been violated. That sounds like a question from recursive function theory; and I think the analogy is sound, remote though it seems from ordinary business practice. An analogue in contracts to the endless passage from n to $n+1$ that gives rise to arithmetic may be found in this remark: if a contract C_n (between A and B, say) could affect the prospects of the players, then so could a contract C_{n+1}! (between A and C) binding its parties to enter no contract of the form C_n. We may hope that the analogy is unsound. At any rate, the apparatus here proposed gives it no footing.

This model admits those and only those contracts C such that, in consideration of certain side payments x among the set S of parties to C, all players in S agree (1) to relinquish all their turns to move in the game tree to a designated agent who is to play a specified mixed strategy σ and (2) to sign no other contract; any member i of S violating (1) or (2) is to pay a specified vector indemnity y^i to the rest of S. Enforcement is paternalistic; without plaintiff, defendant, or trial, the court will assess the indemnities provided for.

Time taken to move or to negotiate is customarily ignored in game theory and can be ignored here, as long as we have three successive periods: the first for general negotiation, the second for an attempt for all players to agree on an outcome, the third (if necessary) for play and assessment and payment of indennities. There is nothing to add to what has been said on the first period, and almost nothing on the third; but the second period bargaining requires a determination of the outcome if no agreement is reached. Now the players may be legally committed to play certain strategies, and may even have inconsistent commitments. In the second period I want them actually committed to a certain course of play. Thus the second period begins wi h each player handing to a clerk a statement of a mixed strategy on the following finite set: the union of his set of pure strategies in the game tree and his set of contracts. (Playing a contract means complying with its play clause, and that would determine his play completely. This "early" commitment to a strategy is objectionable if one thinks of it as early, but the players are supposed to be through negotiating, ready to play, only making a last attempt to secure a jointly optimal outcome by compromise.) Then if the second period results in agreement on an outcome, that is the outcome; if not, play is governed by the strategies given to the clerk.

One remark seems needed, on cancelling contracts. Formally, we allowed the set of all players to cancel or render irrelevant previous contracts. In effect any superset of the set of parties to a contract C can cancel C by drawing up a new contract D (a direct violation of C) and offsetting the indemnities associated with C by the side payments associated with D.

The model is now completely specified, by the second and third paragraphs before this one. It is not specifically a bargaining model, though because of the rudimentary contract law it is a defective model for play. I remark that the customary model for cooperative play, from [7] on, amounts to the first period of this model, with a contract law about a millennium more primitive, followed by play.

Assuming Zeuthen-Nash-Harsanyi bargaining theory, any set S of players can secure jointly in this model $v^-(S)$. (Since the same will hold for the complement of S, it will follow that $v^-(S)$ is all that S can secure.) After the completely trivial proof, we shall note how the model leads to a result so different from the result of Harsanyi's similar model in [3]. Proof: our (second period) bargaining problem, for any threat payoff $(t_1, \ldots, t_n)$, has the solution $(t_1 + a, \ldots, t_n + a)$, where na is the excess $m - \Sigma t_i$[1].Thus the first-period problem for S is(1) the maximin problem, familiar from the Neumann-Morgenstern theory, for the upraised game Γ^-; and its solution is the solution $v^-(S)$ of that problem [7].

How? Well, Harsanyi does not permit S to choose the best coordinated mixed strategy σ and still act as a number of separate persons in the bargaining. *He does not even permit them to play uncoordinated strategies and bargain as separate persons.* In Harsanyi's model, the number corresponding to $v^-(S)$ (call it $u(S)$) comes from S and its complement choosing the best coordinated strategies and bargaining as two ("corporate") persons. (Strictly speaking, the present model does not permit them to bargain separately or corporately as they may prefer; but it would not matter if it did, since Zeuthen-Nash-Harsanyi bargaining theory for games with transferable utility never makes combination gainful.) Accord ingly the rather strange function u that emerges is not a plausible characteristic function (not superadditive). Harsanyi puts no stress on u and describes his model

(1) The fact that the problem for "S", which is not a legal individual, is fairly simple follows from their abil ity to secure each other's continued cooperation by large indemnities.

as a model for bargaining, not for play. The presentation in [3] goes on, in effect reproducing Harsanyi's previous [2] bargaining-model derivation of the Shapley value.

2. **Crucial examples.** There are two rather simple three-person games, for one of which the contract value seems prima facie sound and the Selten value(2) strange, with the reverse holding for the other. It is not only as a debating tactic that I present the latter first; it is simpler. In fact, there is no need to bother about the details of moves. There are three players, Adams, Brown, Cox. If Adams wishes it, Brown will receive 90 units. Otherwise no one gets anything. Call the game Γ_{90}.

The Selten value is (45, 45, 0). Surely this seems obviously right. But the upraised characteristic function gives Adams (precisely, {Adams}) 30, Brown 30, Cox 0, and so on, and the contract value is (40, 40, 10).

Several objections may be made to this contract value. First, what has Cox to do with the game? (With a little jargon, the objector can give an answer instead of a question; Cox is a *dummy*, and has nothing to do with the game.) The model already described provides an answer. The indicated contract for Cox to try to negotiate with (e.g.) Adams will call for Cox to pay Adams 15 units, and for Adams to deny Brown the 90, paying Cox a prohibitively large indemnity if he violates. If this is legal, and accomplished, Brown cannot afterward hope to split 45–45 with Adams; Cox has bought a full one-third share in the enterprise of stealing 90 dollars, or whatever it is. Cox can expect a gross return of 30, net of 15. The reader can easily work out the whole analysis.

There is a second-order objection that I have met often enough to justify a comment on it. "Why don't Adams and Brown close ranks and get 45 instead 40?" They may. Or Adams and Cox may close ranks. That would get Adams 45.

The second serious objection is, if Cox can do this, what about Davis? We were discussing a three-person game. It seems to me that the reminder that the enterprise might consist of stealing 90 dollars almost suffices to answer this. It is notorious that in applying game theory to the description of actual conflict situations, often the hardest part is to say what game is being played.

There is a legitimate question, what happens if there are very many players in the situation of Cox. Some answers are given in Section 3 below.

Of course there are other legitimate questions on various levels (Is this model for cooperative play socially useful?); not for this paper.

A counterquestion: how does Selten justify the value (45, 45, 0)? By axioms. A *dummy* is a player having no moves, and the same payoff at every outcome. Two axioms require dummies to get nothing and to exert no influence [9]. Harsanyi's model secures that value by requiring Cox in effect to marry Adams and lose his legal identity if he makes any agreement with Adams at all.

(2) Selten's paper [9] discusses a number of values, but the main results (according to Selten's Introduction) concern this one and its constant — sum specialization.

Presumably some readers are now at least provisionally convinced. I must address readers who consider dummies outside the community or object to a pluralistic society. The next game Γ_{54} has merits for that purpose and will lead to other points too. The moves are as before, but the payoff is (0, 54, 36) if Adams wishes, otherwise (0, 0, 0). Here the contract value is the rather obvious (30, 30, 30). The reader can check the analysis, but a reader who stayed with Selten for the previous example will give no credit to the contract value. An ad hoc argument for (30, 30, 30): each player may be supposed to ask for a "fair share". Adams' claim is ironclad; without him, nothing can be done. Brown's claim is at least as good as Cox's. Cox's claim is that the other players cannot get anything without giving Cox more than 30; if they wish to divide 90, they can do it foolishly and overpay Cox, or they can treat it as a project requiring unanimous agreement and share equally.

The Selten value is (33, 33, 24). All arguments for it (that I know of) depend somehow, and I think must depend, on a judgment that the 54–36 security which Adams-Brown have against irrational behavior, or skillful bargaining, or other deviant actions by Cox, is worth something, while the lower-level security which Adams-Cox have is worth nothing. (If it were worth something, Adams would have a higher value than Brown.) Of course the dividing line is 45, and an unfriendly critic can readily explain it as the product of a naive theory of coalition formation.

I do not see how the friendliest critic can maintain the (33, 33, 24) value in a social context like that of Harsanyi's model(3) or my more detailed model in Section 1. If the players are bargaining, we may suppose Cox to say: "Your insurance against my deviations is interesting, and I wish I had insurance against your theory. I will not accept 24. If you wish to optimize, give me 30. If you value your insurance more, take it, and I will pocket my 36".

If the context is arbitration, Cox may be more helpless, but otherwise his argument still seems sound. The more serious defense of (33, 33, 24), in line with the insurance idea, treats the value as some sort of average outcome of play. (With the best precedent; the idea is older [10] than the idea of a value as outcome of a definite procedure.) It is difficult to criticize a distribution of which only the mean is known, but we can look.

Since an insurance policy paying 45 is worth 0 in this setting, the last 9 of Adams-Brown's 54 provides all their advantage. (We are comparing Γ_{54} with Γ_{45}, defined in the obvious way. The Selten and contract values for Γ_{45} agree: (30, 30, 30).) This is $4\frac{1}{2}$ units each, some of the time. Their gain in value is 3 each, in the mean. Clearly we cannot get such a result by giving Adams and Brown the extra $4\frac{1}{2}$ only the one-third of the time that the A–B coalition may be supposed to form.

(3) The working of Harsanyi's model for Γ involves three shotgun marriages, i.e. coalitions formed in order to lose.

We must assume that if Cox combines with Adams, he will have to compensate Adams for entering the less secure *A–C* partnership instead of *A–B*(4). This suggests that Adams can be $4\frac{1}{2}$ better off two-thirds of the time. (For if Brown-Cox combine, the insurance available for Adams-Brown cannot help Adams.) But the possibility of Adams-Brown getting 60 instead of 54 has completely disappeared. *The player excluded from a two-man coalition must behave deviantly with probability* 1. And with invariable success.

Perhaps a better explanation, though not for supporting Selten's theory, is that as this game is 20% of the way from Γ_{45} to Γ_{90}, where Cox would be a dummy, Cox is suffering from 20% leprosy. There is a serious point here; is Selten's value for Γ_{54} required by linearity to be the weighted average that it is of the values of Γ_{45} and of Γ_{90}? In a way, yes; the value of Γ_{54} is determined by the axioms. But the linearity notion involved in the axioms, a standard one, does not make Γ_{54} a weighted average of the other two.

$.8\,\Gamma_{45} + .2\,\Gamma_{90}$ is a well-defined game *M*, played as follows. There is an initial chance move, after which either Γ_{45} (with probability .8) or Γ_{90} (with probability .2) will be played. Selten's axioms require *M* to have the corresponding average value, (33, 33, 24), which happens to be the same as that of Γ_{54}. The contract value also has this linearity property, and the contract value of *M* is (32, 32, 26). It differs from the Selten value just in proportion as the contract value of Γ_{90} differs from the Selten value of Γ_{90}. But *M* is not at all the same as Γ_{54}; the difference has some value according to my theory, none according to Selten's. The difference is just that the one windfall of 90, which in Γ_{54} Adams can give or withhold, breaks into two parts in *M*. The parts happen to be mathematical expectations, but for game theory it would be the same if they were certainties, mere physical parts of the 90-unit pile. Adams can give 72, 36 each to Brown and to Cox; and he can give Brown 18 independently of what he does with the 72. Obviously Cox has no claim to equal standing in the game *M*. According to the Selten value, the detached 18 in *M* as compared with Γ_{54} does not strengthen Adams and Brown.

A remark is in order about linearity. The contract value has the property, and I have no argument against linearity; but neither do I regard it as a satisfactory axiom to support dubious theories with. If one drops the artificial hypothesis of transferable utility (greatly increasing the difficulties), neither the Nash cooperative value for two-person games [6], the value for two-person games in my thesis [4], nor any value that I know of satisfies any substantial remnant of linearity. One would not expect equality, for the average of values need not be Pareto optimal in the average of games. But one might expect an inequality. But it fails.

(4) $4\frac{1}{2}$ is enough compensation, for Adams has no prospect in Γ_{54} more than $4\frac{1}{2}$ better than in Γ_{45}.

How does the Selten value of Γ_{54} follow from the axioms? The complete answer is very complicated (fourteen auxiliary games in Selten's proof [9]), but the qualitative point that Cox's value will be less than the others is easily established; and the argument brings out another feature of the theory that may enrich our dossier. Consider the game Δ in which Adams chooses (0, 0, 0) or (−36, 0, −36). In $\Gamma_{54} + \Delta$ (a careful reader may translate via $.5\,\Gamma_{54} + .5\,\Delta$, for correctness) one strategy for Adams yields (−36, 54, 0); so Adams-Brown can certainly secure (9, 9, 0), and Cox is clearly weaker. (It is interesting to compare $\Gamma_{54} + \Delta$ and M; my theory agrees with Selten's that Cox is weaker in these games, but makes him quite a bit worse off in $\Gamma_{54} + \Delta$. But this is not the place for fine points.) Of course that settles Γ_{54}, for the Selten value of Δ is (0, 0, 0).

Look at what the dummy taboo does in Δ. The contract value is (4, 4, − 8). The mechanism is plain; this is a game of extortion, the threat (−36, 0, −36) is not very persuasive, but a threat of (−18, −18, −36) may move Cox. Selten forbids it — this may be socially desirable. It is not impartial. From Adams' point of view as well as Brown's, it prohibits best play.

3. **Residual value.** If Γ has players $1, \ldots, n$, let $D_k(\Gamma)$ denote an $n+k$-player game constructed from Γ by adding dummy players $n+1, \ldots, n+k$, each of whom gets 0 at each outcome of Γ. $D_k\phi^-(\Gamma)$ is the n-vector consisting of the first n components of $\phi^- D_k(\Gamma)$. The *residual value* $\phi^r(\Gamma)$ is $\lim_{k\to\infty} D_k\phi^-(\Gamma)$.

The question of the easiest proof that ϕ^r exists seems mildly interesting; as we see from the example Δ, the vectors $D_k\phi^-(\Gamma)$ need not be monotone decreasing. Here we shall describe a computation previously done by L. S. Shapley (unpublished) and indicate why it yields the residual value. Recall that the Shapley value $\phi_i(D_k(\Gamma)^-)$ of the upraised game with k dummies, to the i-th player, is the average of the expressions $v^-(S \cup \{i\}) - v^-(S)$, averaged over the $(n+k)!$ total orderings of the players, S being the set of predecessors of i. Let $x_j (j = 1, \ldots, n)$ be the fraction of all the players preceding the j-th, $S_0 = \{j \in S : j \leqq n\} = S_0(x, i)$. Then $v^-(S \cup \{i\}) - v^-(S) = \delta_k(x, i)$ is determined by the n-vector x and the indices i, k; $v^-(S)$, for example, is the value of a two-player constant-sum game with sum m derived from Γ by aggregating players, S_0 making the first player and getting at each outcome z the payoff $\Sigma[h_i(z)\colon\ i \in S_0] + x_i(m - \Sigma h_j(z))$. The other term involves k. So

$$\phi_i(D_k(\Gamma)^-) = \int \delta_k(x, i)\, d\mu_k, \tag{3.0}$$

where μ_k is a suitable atomic measure on the unit n-cube. The formulas define $\delta_k(x, i)$ on the whole cube, piecewise uniformly continuous on a fixed set of $n!$ pieces; then so is $\delta(x, i) = \lim_{k\to\infty} \delta_k(x, i)$, for the convergence is uniform. We conclude

$$\phi_i^r = \int \delta(x, i) d\mu, \tag{3.1}$$

where μ is ordinary volume. For the functionals $\int \quad d\mu_k$ (Riemann sums) converge to $\int \quad d\mu$.

To illustrate (3.1), the double integral for ϕ_a^r (a = Adams) in the example Δ reduces to $2 \int_0^{.5} (36x - 72x^2)dx = 3$; ϕ_c^r is -15.

Evidently the operators D_k, ϕ^r are positive-linear on games to games resp. vectors. From this we can get an interesting remark and a computational shortcut. The *constant-sum extension* of Γ, with any chosen constant sum c, is constructed [7] by adjoining a player who gets $c - \Sigma h_i(z)$ at each outcome z. In general the contract value of the constant-sum extension differs from the residual value; for Δ it is $(12, 0, -24, 12 + c)$. But:

(3.2) *The constant-sum extension gives the residual value for fixed-threat games.*

A fixed-threat, or "characteristic function" game is determined by giving a superadditive function v on the set of all sets of players. To play, each player names a set; those sets S named by all their members get $v(S)$ (divided equally among them, say), and players i left out get $v(\{i\})$. Now the characteristic functions, as set functions, are linearly generated [10] by the pure bargaining games B_S determined by $v(T) = 1$ for $T \supseteq S$, $v(T) = 0$ otherwise. Thus every fixed-threat game Γ satisfies a relation $\Gamma + \Sigma\lambda_S B_S = \Sigma\mu_S B_S$ with non-negative coefficients λ_S and μ_S. (Strictly, + for games complicates the strategies; to justify the equation we must modify the sums of games by requiring a player to name the same S for each summand. Clearly this will not affect the values.) So it suffices to prove (3.2) for the games B_S. In the constant-sum extension of B_S with sum 1, the added player's value is the probability that in a total ordering he is between two members of S; this is $(s - 1)/(s + 1)$. The remaining $2/(s + 1)$ is shared equally by the members of S. In $D_k(B_S)$, if the players are totally ordered, each player i of S has $\delta_k(x, i) < 1/k$ except for the first and last of them, who get together nearly the fraction of players not between two members of S. The expected value is again $2/(s + 1)$, and S shares it equally.

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